Hyperbolic Functions, Logarithms of Complex Numbers

Circular functions of a Complex Variable

Euler's formulae :- $e^{ix} = \cos\theta + i\sin\theta$

$e^{-ix} = \cos\theta - i\sin\theta$

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$; $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

↓

Circular function for real x.

Circular functions of Complex Variable

$z = x + iy$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\tan z = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

$\cot z = \dfrac{\cos z}{\sin z} = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

$\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz} + e^{-iz}}$

$\operatorname{cosec} z = \dfrac{1}{\sin z} = \dfrac{2i}{e^{iz} - e^{-iz}}$

Hyperbolic Functions

$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

$$\coth hx = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$\sinh(-x) = -\sinh x$ & $\cosh(-x) = \cosh x$

↓ odd funⁿ of x ↓ even funⁿ of x.

Some Important Formulae

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

x	$\sinh x$	$\cosh x$	$\tanh x$
$-\infty$	$-\infty$	∞	-1
0	0	1	0
∞	∞	∞	1

Relation between Circular and Hyperbolic functions :-

$\sin(ix) = i \sinh x$ $\cot(ix) = -i \coth x$

$\cos(ix) = \cosh x$ $\sec(ix) = \operatorname{sech} x$

$\tan(ix) = i \tanh x$ $\operatorname{cosec}(ix) = -i \operatorname{cosech} x$

Formulae of Hyperbolic functions :-

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\text{sech}^2 x + \tanh^2 x = 1$

(3) $\coth^2 x - \text{cosech}^2 x = 1$

(4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \cdot \sinh y$

(5) $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$

(6) $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$

(7) $\sinh 2x = 2\sinh x \cosh x$

(8) $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right) \cdot \cosh\left(\frac{x-y}{2}\right)$

(9) $\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right) \cdot \sinh\left(\frac{x-y}{2}\right)$

(10) $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right) \cdot \cosh\left(\frac{x-y}{2}\right)$

(11) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right) \cdot \sinh\left(\frac{x-y}{2}\right)$

(12) $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \cdot \tanh y}$

(13) $\sinh x = \frac{2\tanh\frac{x}{2}}{1 - \tanh^2\frac{x}{2}}$

(14) $\cosh x = \frac{1 + \tanh^2\frac{x}{2}}{1 - \tanh^2\frac{x}{2}}$

(15) $\tanh x = \frac{2\tanh\frac{x}{2}}{1 + \tanh^2\frac{x}{2}}$

(16) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(17) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(18) $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

(19) $\frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x$

(20) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(21) $\int \cosh x \, dx = \sinh x$

(22) $\int \sinh x \, dx = \cosh x$

(23) $\int \operatorname{sech}^2 x \, dx = \tanh x$

Inverse ~~trigo~~ Hyperbolic function :-

(24) $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

(25) $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

(26) $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(27) $\sinh^{-1} x = \cosh^{-1} \sqrt{1+x^2}$

(28) $\sinh^{-1} x = \tanh^{-1} \frac{x}{\sqrt{1+x^2}}$